# ANNALS OF MATHEMATICS.

VOL. IV. FEBRUARY, 1888. No. 1.


## A SOLUTION OF KEPLER'S PROBLEM FOR PLANETARY ORBITS OF HIGH ECCENTRICITY.


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The solution of Kepler's Problem involves the transcendental equation

$$M = E - e \sin E, \tag{1}$$

in which $M$ and $E$ are respectively the mean and eccentric anomalies, while $e$ is the eccentricity.

Let

$$\begin{aligned} F(E) &= \tfrac{1}{2}[E - M - \sin(E - M)] \\ &= \tfrac{1}{12}\sin^3(E-M) + \tfrac{3}{80}\sin^5(E-M) \\ &\quad + \tfrac{5}{224}\sin^7(E-M) + \ldots \end{aligned} \tag{2}$$

From (1) and (2) we easily get Grunert's equation,

$$\frac{1+e}{1-e} = \frac{\cos\frac{1}{2}M\sin(E-\frac{1}{2}M) + F(E)}{\sin\frac{1}{2}M\cos(E-\frac{1}{2}M) - F(E)}. \tag{3}$$

Equation (3) gives

$$\tan(E - \tfrac{1}{2}M) = \frac{1+e}{1-e}\tan\tfrac{1}{2}M - \frac{2F(E)}{(1-e)\cos\frac{1}{2}M\cos(E-\frac{1}{2}M)}. \tag{4}$$

Assume that

$$\tan(E' - \tfrac{1}{2}M) = \frac{1+e}{1-e}\tan\tfrac{1}{2}M. \tag{5}$$

Substituting in (4) according to (5) and reducing, we obtain

$$\sin(E'-E) = \frac{\cos(E'-\frac{1}{2}M)}{(1-e)\cos\frac{1}{2}M}\,.\,2F(E) \tag{6}$$

$$= k\,.\,2F(E). \tag{7}$$

Hence for moderate eccentricities, $E' - E$ is approximately equal to $2F(E)$.

Let

$$2\eta = E' - M - \sin(E' - M). \tag{8}$$

Then $E' - E$ is likewise nearly equal to $2\eta$, and $E' - M - 2\eta$ is nearly equal

to $E - M$. From (2) we obtain by differentiation

$$dF(E) = [\tfrac{1}{4}\sin^2(E-M) + \tfrac{3}{16}\sin^4(E-M) + \ldots]\cos(E-M)\,d(E-M). \quad (9)$$

Equation (9) shows that a small change in the value of $E - M$ produces a much smaller change in $F(E)$; therefore, we write with but slight error

$$2\,F(E) = E' - M - 2\eta - \sin(E' - M - 2\eta). \quad (10)$$

From (8), omitting terms of the seventh and higher orders, we have

$$\eta = \tfrac{1}{12}\sin^3(E' - M) + \tfrac{3}{80}\sin^5(E' - M). \quad (11)$$

Suppose that $E'$ is changed by an amount equal to $2\eta$; i. e. that $dE' = 2\eta$. Differentiation of (11) and reduction of the resulting equation, neglecting powers of $\sin(E' - M)$ higher than the fifth, gives

$$d\eta = \tfrac{1}{24}\sin^5(E' - M)\cos(E' - M); \quad (12)$$

$$\therefore\ \eta - d\eta = \tfrac{1}{12}\sin^3(E' - M) + \tfrac{3}{80}\sin^5(E' - M) - \tfrac{1}{24}\sin^5(E' - M)\cos(E' - M). \quad (13)$$

The last two terms of (13) have nearly the same magnitude, and opposite signs. We therefore write

$$\eta - d\eta = \tfrac{1}{12}\sin^3(E' - M). \quad (14)$$

Now since (12) is obtained on the assumption that $dE' = 2\eta$,

$$\eta - d\eta = \tfrac{1}{2}[E' - M - 2\eta - \sin(E' - M - 2\eta)].$$

Hence by (10) and (14) we may write

$$F(E) = \tfrac{1}{12}\sin^3(E' - M). \quad (15)$$

Equations (15) and (7) give

$$\sin(E' - E) = \tfrac{1}{6}k\sin^3(E' - M). \quad (16)$$

Let $E' - M - \frac{1}{6}k\sin^3(E' - M)$ be an approximate value of $E - M$; then $2F[E' - M - \frac{1}{6}k\sin^3(E' - M)]$ will be the corresponding value of $2F(E)$, and by (7), $k\,.\,2F[E' - M - \frac{1}{6}k\sin^3(E' - M)]$ will be the corresponding value of $\sin(E' - E)$. Differentiation of (7) gives

$$\begin{aligned}\cos(E' - E)\,d(E' - E) &= k\,[\tfrac{1}{2}\sin^2(E - M) + \tfrac{3}{8}\sin^4(E - M) + \ldots]\cos(E - M)\,d(E - M)\\ &= k\,[\tfrac{1}{2}\sin^2(E - M) + \tfrac{1}{8}\sin^4(E - M) + \ldots]\,d(E - M)\\ &= kv\,d(E - M). \quad (17)\end{aligned}$$

If $E - M$ be increased by $E' - E - \frac{1}{6}k\sin^3(E' - M)$, and the corresponding increment of $E' - E$ be denoted by $\Delta(E' - E)$, we have

$$\Delta(E' - E) = kv\,[E' - E - \tfrac{1}{6}k\sin^3(E' - M)]\sec(E' - E). \quad (18)$$

But

$$E' - E + \Delta(E' - E) = k\,.\,2F[E' - M - \tfrac{1}{6}k\sin^3(E' - M)] + 2F(E' - E). \quad (19)$$

Substitute from (18) in (19), and add and substract $\frac{1}{6}k \sin^3(E' - M)$ in the second member of the resulting equation; reduction gives

$$\begin{aligned}(E' - E)[1 + kv \sec(E' - E)] = [1 + kv \sec(E' - E)] \,.\, \tfrac{1}{6}k \sin^3(E' - M) \\ + k \,.\, 2F[E' - M - \tfrac{1}{6}k \sin^3(E' - M)] \\ - \tfrac{1}{6}k \sin^3(E' - M) + 2F(E' - E). \qquad (20)\end{aligned}$$

Division of (20) by the coefficient of $E' - E$, and subtraction of $2F(E' - E)$ from each side, gives

$$\begin{aligned}\sin(E' - E) = \tfrac{1}{6}k \sin^3(E' - M) \\ + \frac{k\{2F[E' - M - \frac{1}{6}k \sin^3(E' - M)] - \frac{1}{6}\sin^3(E' - M)\}}{1 + kv \sec(E' - E)} \\ \frac{kv \sec(E' - E) \,.\, 2F(E' - E)}{1 + kv \sec(E' - E)} \qquad (21)\end{aligned}$$

But we have, with sufficient accuracy,

$$2F\,(E' - E) = \tfrac{1}{6} \sin^3(E' - E) = \tfrac{1}{1296}k^3 \sin^9(E' - M),$$

and

$$\frac{kv \sec(E' - E) \,.\, 2F(E' - E)}{1 + kv \sec(E' - E)} = \frac{k^4 \sin^{11}(E' - M)}{2592\,(1 + kv)}. \qquad (22)$$

We next determine the error of (21) when its last term is neglected and the quantity $\sec(E' - E)$ is dropped from the denominator of the second fraction. The eccentricities of the orbits of all but 19 of the first 268 asteroids are below sin 15°. Of these 19, 15 lie between sin 15° and sin 20°, and 4 are greater than sin 20°; the orbit of Aethra (132) has the greatest eccentricity, $e$ being equal to sin 22°.5. On the assumption that $e = \sin 23°$, we find that the maximum value of the last term of (21), as computed by (22), is 0″.003 +. If the factor $\sec(E' - E)$ in the denominator of the second fraction of (21) be dropped, an error of 0″.0002 may result. To compute $v$ in the same denominator, we substitute for $E - M$ (in the expression given for $v$ in (17)) the value $E' - M - \frac{1}{6}k \sin^3(E' - M)$. From equations (17)–(21) we see that, had $E' - E - \frac{1}{6}k \sin^3(E' - M)$, the increment of $E - M$, been *infinitesimal*, $E' - M - \frac{1}{6}k \sin^3(E' - M)$ might have been used for $E - M$ in computing $v$, without error. But since $E' - E - \frac{1}{6}k \sin^3(E' - M)$ is a small finite quantity, the proper value of $E - M$ to employ is

$$E' - M - \tfrac{1}{6}k \sin^3(E' - M) - \tfrac{1}{2}[E' - E - \tfrac{1}{6}k \sin^3(E' - M)].$$

The greatest error in $E' - E$ caused by using $E' - M - \frac{1}{6}k \sin^3(E' - M)$ for $E - M$, in finding $v$, is less than 0″.002. Greater accuracy is unnecessary, but could be attained, since $1 + kv$ is always close to unity, by using

$$\begin{aligned}E' - M - \tfrac{1}{6}k \sin^3(E' - M) \\ - \tfrac{1}{2}k\{2F[E' - M - \tfrac{1}{6}k \sin^3(E' - M)] - \tfrac{1}{6}\sin^3(E' - M)\},\end{aligned}$$

or $\quad E' - M - \frac{1}{12}k \sin^3 (E' - M) - k\,F[E' - M - \frac{1}{6}k \sin^3 (E' - M)].$

Equation (21) may therefore be written

$$\sin (E' - E) = \frac{1}{6}k \sin^3 (E' - M) + \frac{k\,\{2F[E' - M - \frac{1}{6}k \sin^3 (E' - M)] - \frac{1}{6} \sin^3 (E' - M)\}}{1 + kv}. \quad (23)$$

When $e = \sin 23°$, the maximum numerical value of the last fraction of (23) is $38''.6$. A simple approximate value of this fraction in terms of $k$ and sin $(E' - M)$ may be obtained as follows:—

From (9) and (2), by putting $E' - M$ for $E - M$, $\frac{1}{6}k \sin^3 (E' - M)$ for $d(E - M)$, and $1 - \frac{1}{2} \sin^2 (E' - M)$ for $\cos (E' - M)$, we get

$$2F[E' - M - \frac{1}{6}k \sin^3 (E' - M)] = \frac{1}{6} \sin^3 (E' - M) + (\tfrac{3}{40} - \tfrac{1}{12}k) \sin^5 (E' - M) + (\tfrac{5}{112} - \tfrac{1}{48}k) \sin^7 (E' - M). \quad (24)$$

Since $kv$ is small, we assume that

$$\frac{1}{1 + kv} = 1 - kv = 1 - \frac{1}{2}k \sin^2 (E' - M). \quad (25)$$

Substitution of (24) and (25) in the last fraction of (23) gives for its value

$$\tfrac{1}{120}(9k - 10k^2) \sin^5 (E' - M) + [\tfrac{1}{336}(15k - 7k^2) - \tfrac{1}{240}(9k^2 - 10k^3)] \sin^7 (E' - M)$$

Since $\sin (E' - M) = e \sin E'$, neglecting powers higher than the fifth, the preceding expression becomes $\frac{1}{120} (9k - 10k^2)\, e^5 \sin^5 E'$. Because this involves the fifth power of $e$, its value is small for moderate eccentricities; nevertheless, the expression is not sufficiently accurate to be employed in seven-place computations with large eccentricities. As the second member of (23) is not easy to compute, it may be simplified, and its errors tabulated. It is not difficult to show that the maximum value of $k$ is $(1 - e)^{-1}$, which is reached when $v = 0$. The maximum of $kv$ is near $\frac{1}{2}e^2$, which is only 0.08 when $e = \sin 23°$; $kv$ being, therefore, always small, neglect of the denominator of the last term of (23) introduces into $E' - E$ an error the value of which is not far from

$$\tfrac{1}{120}\, kv\, (9k - 10k^2)\, e^5 \sin^5 E', \text{ or } \tfrac{1}{240}\, (9k^2 - 10k^3)\, e^7 \sin^7 E'.$$

When $e = \sin 23°$ this error may amount to $3''$; when $e = \sin 15°$ it does not reach $0''.2$. Rejecting $1 + kv$ and writing $E' - E$ for $\sin (E' - E)$, (23) becomes

$$E' - E = k\,.\,2F[E' - M - \tfrac{1}{6}k \sin^3 (E' - M)] \quad (26)$$

To expedite computation, two tables are needed, one giving the correction of the second member of (26), the other containing $2F(x)$ with the argument $x$. The computer having $e$ and $M$ given, would use (5) and (26), taking into account the tabulated correction to the value of $E' - E$ given by (26). From the value of $E$ thus found, the true anomaly could be obtained by one of the usual methods.